Real-fibered morphisms of real del Pezzo surfaces

Joint with Mario Kummer and Cédric Le Texier

$X/_{\mathbb{C}}$ non-singular alg varieties

$\dim X = n$

$\sigma : X \to X$ real structure

$\mathrm{fix}(\sigma) = \mathbb{R}X$ real part

__Def__: $f : \underset{\circlearrowleft \sigma}{X} \to \underset{\circlearrowleft \mathrm{conj}}{\mathbb{CP}^n}$ real

morphism, f is __real-fibered__

if $f^{-1}(\mathbb{RP}^n) = \mathbb{R}X$

- (X, σ) $\quad \mathbb{R}X \neq \emptyset$

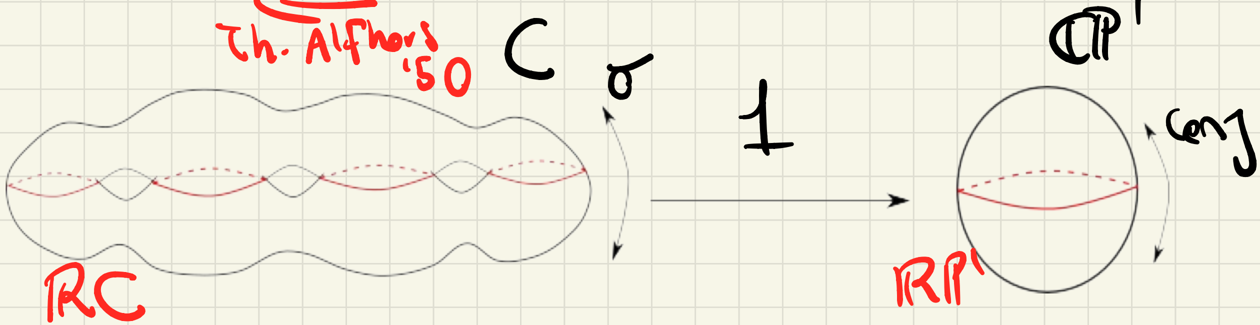
$\boxed{n=1}$ (C, σ) real curves

$\exists\, f: C \to \mathbb{CP}^1$ real fibered	$C \setminus \mathbb{R}C$ is disconnected (SEPARATING)	$[\mathbb{R}C]$ trivial in $H_1(C; \mathbb{Z}/2\mathbb{Z})$

$\exists\, f: C \to \mathbb{CP}^1$ real fibered $\iff$ $C \setminus \mathbb{R}C$ is disconnected (SEPARATING) $\iff$ $[\mathbb{R}C]$ trivial in $H_1(C; \mathbb{Z}/2\mathbb{Z})$

$\Longrightarrow$ $\mathbb{CP}^1 \setminus \mathbb{RP}^1$ non-connected

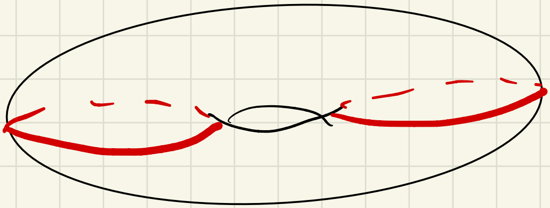
$\Longleftarrow$ Th. Alfhors '50



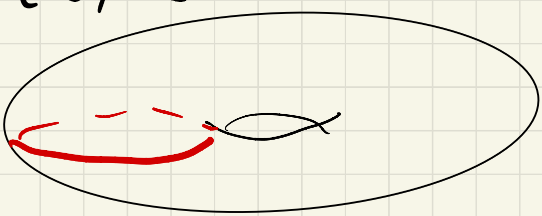
examples

$n=1$ $\quad g=1$

(C,σ_1) $\quad \mathbb{R}C \simeq S^1 \sqcup S^1$



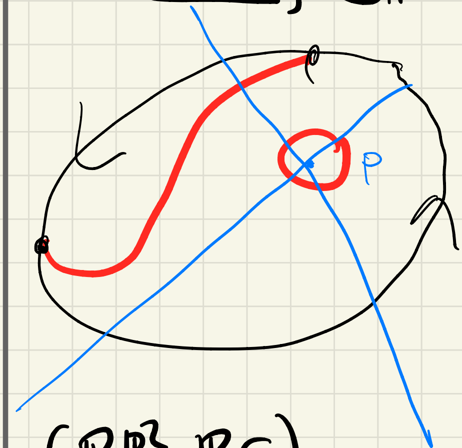
(C,σ_2) $\quad \mathbb{R}C \simeq S^1$



$$H_1(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

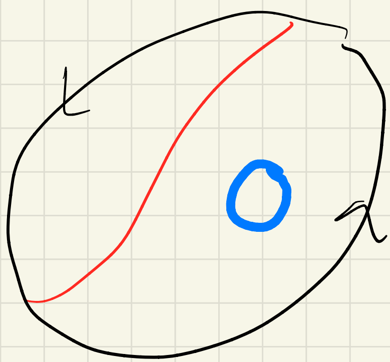
$[A] = 1$

$C \hookrightarrow \mathbb{CP}^2$



$(\mathbb{RP}^2, \mathbb{R}C)$

$\pi_p|_C \; C \to \mathbb{CP}^1$ real fibered



$(\mathbb{RP}^2, \mathbb{R}C)$

- (C, σ) real algebraic separating curve

1) $\boxed{\ell \equiv g(C)+1 \mod 2}$ $\quad \ell = \#$ c.c. of $\mathbb{R}C$

2) $\ell = g(C)+1$ maximum for ℓ $^{\text{H-K}}$
$\rightsquigarrow$ C is <u>MAXIMAL</u>.
(Harnack-Klein inequality)

- Any real alg. maximal curve is separating.

From $m=1$ to $m\geq 2$

$m=1$

$\exists\, f : \underset{\circlearrowleft\,\sigma}{C} \longrightarrow \underset{\circlearrowleft\,\mathrm{conj}}{\mathbb{CP}^1}$ real-fibered $\iff$ $C\setminus\mathbb{R}C$ is $\underbrace{\text{disconnected}}_{\text{SEPARATING}}$ $\iff$ $[\mathbb{R}C]$ is trivial in $H_1(C;\mathbb{Z}/2\mathbb{Z})$

$m\geq 2$

(X,σ)
$\exists f : X \to \mathbb{CP}^m$ real fibered?

$\not\Longleftrightarrow$

VIRO '94
(X,σ) bounds in complexification
$[\mathbb{R}X]$ in $H_m(X;\mathbb{Z}/2\mathbb{Z})$ trivial

(X, σ) VIRO $\not\Longrightarrow$ $\exists\, f : X \to \mathbb{CP}^n$ red-fib.

$$\not\Longleftarrow$$

$\boxed{n \geq 2}$ RIGIDITY

(X,τ) $\exists f : X \longrightarrow \mathbb{CP}^n$ real fibered

Kummer-Shamovic '15:

$f|_{\mathbb{R}X} : \mathbb{R}X \longrightarrow \mathbb{RP}^n$ unramified

(1) $\mathbb{R}X = \bigsqcup_{s} S^n \sqcup \bigsqcup_{r} \mathbb{RP}^n$

(2) $\deg f = 2s + r$

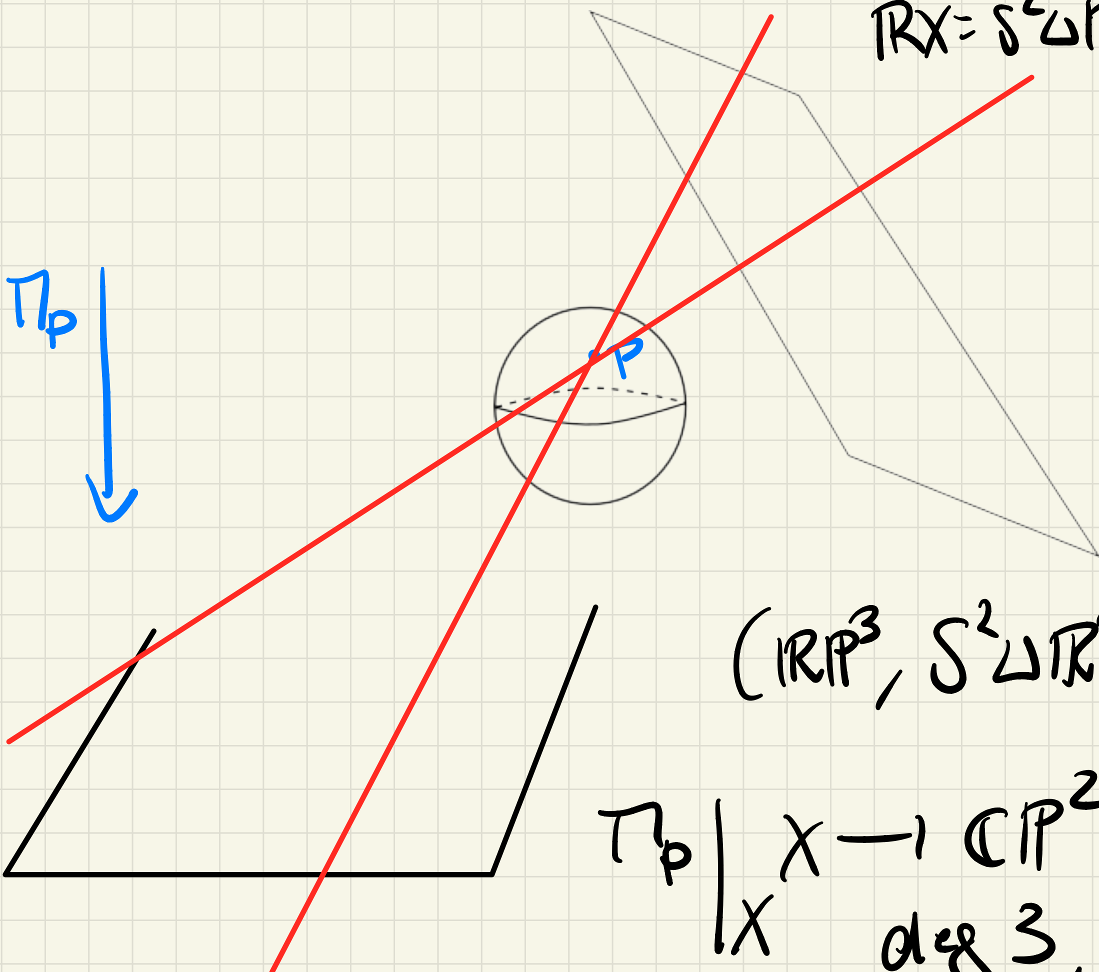
Rk: (X,σ) $\exists f : X^{p_1,p_2} \longrightarrow \mathbb{CP}^n$ real fibered
$\circlearrowleft \sigma$ $\quad \circlearrowleft conj$

$\tilde{f} : Bl_{p_1,p_2}(X) \longrightarrow \mathbb{CP}^n$ real fibered non-finite

p_1, p_2 cx conj

<u>example</u>: X cubic surface $\subset \mathbb{CP}^3$

$\mathbb{R}X = S^2 \sqcup \mathbb{RP}^2$



$(\mathbb{RP}^3, S^2 \sqcup \mathbb{RP}^2)$

$\pi_p\big|_X : X \to \mathbb{CP}^2$

deg 3

real-fibered

DEF: $f: X \to \mathbb{CP}^m$ real fibered

if $f: X \overset{i}{\hookrightarrow} \mathbb{CP}^n \overset{\pi_E}{\dashrightarrow} \mathbb{CP}^m$

f is <u>hyperbolic</u> wrt E

X is <u>hyperbolic</u>

E codim $m+1$ in $\mathbb{CP}^n$ $E \cap X = \emptyset$

(D divisor on X real very ample

look for E $\pi_E|_X\ X \to \mathbb{CP}^m$ real fibered)

<u>Remark</u>: $\boxed{m=1}$ <u>Kummer - Shaw '18</u>

- Any separating curve admits an hyperbolic morphism.

- There are real-fibered morphs. which are not hyperbolic.

Del Pezzo surfaces

DEF: X is a non-sing. alg surface is del Pezzo if $-K_X$ ample

$$\deg X = (-K_X)^2 = d \qquad 0 \leq d \leq 9$$

CX PT OF VIEW: $Bl_{p_1 \dots p_r}(\mathbb{CP}^2) = X$

$r = 0 \; \dots \; 9$

- (X, σ) class of real del Pezzo surfaces is known (COMESSATTI '13 '14)
- $\exists \, (X, \sigma)$ $\quad \mathbb{R}X = \bigsqcup_s S^2 \sqcup \bigsqcup_r \mathbb{RP}^2$

THM (KLT+M): Let (X,σ) real DP surface with $\mathbb{R}X = \bigsqcup_s S^2 \sqcup \bigsqcup_r \mathbb{RP}^2$

$\exists$ FINITE REAL FIBERED (hyperbolic)

$f: X \longrightarrow \mathbb{CP}^2$ if one has

(1) X has real Picard rank 1

or

(2) X is a conic bundle with real Picard rank 2

or

(3) X is the blow up at one or ~~two~~ real points of one of the above surfaces.

Examples:

- cubic surface previous example

- (non-hyperbolic case)

(X, σ) $\deg X = 2 = (-K_X)^2$

$\mathbb{R}X = \mathbb{RP}^2 \sqcup \mathbb{RP}^2$

$$X \xrightarrow{2-1} \mathbb{CP}^2 \hookleftarrow Q$$

real plane quartic

$\mathbb{R}Q = \emptyset$

Main ingredients of the proof

(X, σ) del Pezzo Surf. $\mathbb{R}X = \bigsqcup_s S^2 \sqcup \bigsqcup_r \mathbb{RP}^2$
admits a finite $f : X \longrightarrow \mathbb{CP}^2$
real fibered $\rightsquigarrow$ ample divisor D on X

- $\max\{1, r\} \leq D \cdot K_X + 4 \leq 2s + r$
- $D \cdot K_X \equiv r \mod 4$

$\underline{\mathbb{R}K}$: $f^{-1}(L) = C$ a general line in $\mathbb{CP}^2$ is separating curve in X $s+r$ c.c. in $\mathbb{R}C$

$r + s \equiv g(C) + 1 \mod 2$

COROLLARY: $(X, \overline{\sigma})$ real DP $\exists$ # of candidate real morphisms which may be real fibered

- brute force

Reider

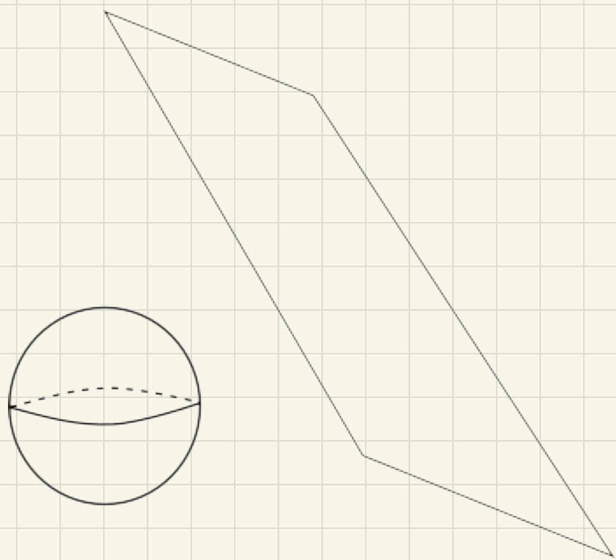
- D very ample and real (di Rocco 96)

- Find $E \subset \mathbb{CP}^4$ s.t. $E \cap X = \emptyset$
  and $\pi_E|_X \colon X \longrightarrow \mathbb{CP}^2$ real-fib.

+ . . .

$X \subset \mathbb{CP}^3$

Back to cubic surface example



$(\mathbb{RP}^3, \mathbb{R}X)$

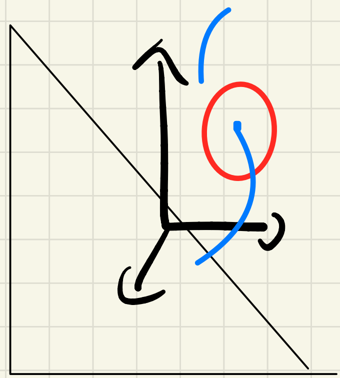
Need of such a criterion

ex of cubic surface:

in $\mathbb{RP}^3$

if $X \subseteq \mathbb{CP}^n$ $n > 3$

$S^2 \subseteq \mathbb{RP}^n$



A way to assure that con constr. a codim 3 lin. sub. E s.t.

$F^{n-2} \supseteq E$ $F^{n-2} \cap X$ in only real pts.

LINKING NUMBERS

$\ell k(S^2, \mathbb{R}E) = \pm 2$ $\ell k(\mathbb{RP}^2, \mathbb{R}E) = \pm 1$

DEF: $K, L \in S^h$ $\quad$ K m-sphere, L ℓ-sphere

$h = m + \ell + 1$

$[K]$, $[L]$ fund. cycles

W chain $\quad \partial W = K$

$\ell k(K, L) = \# W \cap L$

DEF: $X \subseteq \mathbb{CP}^h$ $\quad \mathbb{R}X = X_1 \sqcup \ldots \sqcup X_{s+r}$

$\dim X = m$ $\quad \mathbb{R}X = \bigsqcup_s S^m \sqcup \bigsqcup_r \mathbb{RP}^m$

$E \cap X = \emptyset$

E lin sub. $\subseteq \mathbb{CP}^h$ codim $m+1$

$\pi : S^h \to \mathbb{RP}^h$

$\ell k(X_i, \mathbb{R}E) = \ell k(\pi^{-1}(X_i), \pi^{-1}(\mathbb{R}E))$

$\cap$

c.c. $\mathbb{R}X$

TH(KLTM) 2 :

$X \subseteq \mathbb{CP}^n$, $E \subseteq \mathbb{CP}^n$ as above

$\deg X = 2s + r$. X is hyperbolic wrt E iff

$$\boxed{2s+r = \sum_{i=1}^{s+r} \left| \ell k \left(X_i, \mathbb{R}E \right) \right|}$$

$\boxed{n=1}$ Kummer-Shaw '18

- $\ell k(S^2, \mathbb{R}E) = \pm 2$
- $r = 0,1$ $\quad \ell k(\mathbb{RP}^2, \mathbb{R}E) = \pm 1$

<u>RK</u>: (hyperbolic case for del Pezzo)

- $X \hookrightarrow \mathbb{CP}^h$ $h = s+2$
- $\mathbb{R}X \supseteq$! $\mathbb{RP}^2$ or none

TH(KLTM)3: $X \subset \mathbb{CP}^h$ surface $\mathbb{R}X \simeq \bigsqcup_s S^2 \sqcup \bigsqcup_r \mathbb{RP}^2$ $h = s+2$ $r \in \{0,1\}$. Assume $g(H \cap X)$ is $s+r-1$. $\exists$ E l.s. s.t. X is hyperbolic wrt E.

<u>Proof</u>: $h > s$ can construct a hyperplane H s.t. $C = H \cap X$ is a real separating curve with $s+r$ c.c. in $\mathbb{R}X$.

Each c.c. of $\mathbb{R}X$ contains contains exactly one c.c. of $\mathbb{R}C$ $\xRightarrow{\text{Kummer-Shaw '18}}$ $\exists\, E \subset H$ l.s. s.t.

C is hyperbolic wrt E in H

$\xRightarrow{\text{TH ek \#}}$ X is hyperbolic wrt E in $\mathbb{C}P^n$.